

AN INTRODUCTION TO MULTILINEAR ALGEBRA AND SOME APPLICATIONS

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Abstract

The present paper intends to give a quick and easy introduction to tensors, in particular to the exterior and the symmetric algebra of a vector space and the relations between them. The entries of the k -th exterior, and the k -th symmetric power of a matrix are expressed as determinants, and permanents, respectively. The generating polynomials of these powers are related by their traces. As applications we mention MacDONALD's proof of MacMAHON's master theorem, BEBIANO's permanent expansion¹⁾ and the permanent version of the solution of Fredholm's integral equation, observed by KERSHAW²⁾ and by de BRUIJN³⁾.

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1. Symmetric functions⁴⁾

The *elementary symmetric* polynomial of degree k in d variables, x_1, \dots, x_d , is defined by

$$e_k(\mathbf{x}) := \sum_{1 \leq j_1 < \dots < j_k \leq d} x_{j_1} \dots x_{j_k},$$

or, equivalently, by its generating function

$$E(t; \mathbf{x}) := \sum_{k \geq 0} e_k(\mathbf{x}) t^k = \prod_{i=1}^d (1 + x_i t).$$

The k -th *complete symmetric* polynomial is defined by

$$h_k(\mathbf{x}) := \sum_{1 \leq j_1 \leq \dots \leq j_k \leq d} x_{j_1} \dots x_{j_k},$$

with generating function

$$H(t; \mathbf{x}) := \sum_{k \geq 0} h_k(\mathbf{x}) t^k = \prod_{i=1}^d \frac{1}{1 - x_i t}.$$

Theorem 1.1

$$E(-t; \mathbf{x}) H(t; \mathbf{x}) = 1.$$

Remark

$\{e_1, \dots, e_d\}$ forms a basis for the graded ring S_d of symmetric polynomials

in d variables. Let j denote the all-one vector of size d , then

$$e_k(j) = \binom{d}{k} \quad \text{and} \quad h_k(j) = \binom{d+k-1}{k}.$$

2. Tensors⁵⁻⁷⁾

We start with a quick definition. Let V be a (real) vector space of dimension d , with (positive definite) inner product $(,)$ and orthonormal basis e_1, \dots, e_d . We define tensor products by their components with respect to this basis:

$$\begin{aligned} x \otimes y & \text{ by the } d^2 \text{ components } x_i y_j, \\ x \otimes y \otimes z & \text{ by the } d^3 \text{ components } x_i y_j z_k. \end{aligned}$$

The tensor product $V \otimes V$ is the linear space spanned by all tensor products $x \otimes y; x, y \in V$. The inner product $(x \otimes y, u \otimes v) := (x, u)(y, v)$ provides $V \otimes V$ with the orthonormal basis $\{e_i \otimes e_j \mid 1 \leq i, j \leq d\}$.

Similarly tensor products of k vectors, $x_1 \otimes \dots \otimes x_k$ are defined by their d^k components $(x_1)_i (x_2)_j \dots (x_k)_m$, with respect to the orthonormal basis

$$\{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_m} \mid 1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq d\}$$

of $T_k(V) := \otimes^k V$, with inner product

$$(x_1 \otimes \dots \otimes x_k, y_1 \otimes \dots \otimes y_k) = \prod_{i=1}^k (x_i, y_i).$$

Clearly, $\dim T_k V = d^k$. The tensor algebra TV is the direct sum

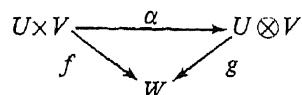
$$TV := T_0 V \oplus T_1 V \oplus \dots = \sum_{k \geq 0}^{\oplus} T_k V,$$

where $T_0 V \simeq R, T_1 V \simeq V$. Its Poincaré polynomial (as a graded algebra) is

$$P_T(t) := \sum_{k \geq 0} (\dim T_k V) t^k = \frac{1}{1 - dt}.$$

Remark

We refer to the literature for details about the following coordinatefree definition. The tensor product of the vector spaces U and V is the pair of a vector space $U \otimes V$ and a bilinear map $\alpha : U \times V \rightarrow U \otimes V$ such that for all bilinear f there is a unique linear g such that the following diagram commutes



3. The exterior algebra $\Lambda V^{5,6)$

$\Lambda_2 V$ is the linear subspace of $T_2 V$ spanned by the skew 2-tensors:

$$x \wedge y := \frac{1}{2}(x \otimes y - y \otimes x); \quad x, y \in V.$$

Clearly, $\mathbf{x} \wedge \mathbf{x} = \mathbf{0}$, $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$ and $\dim \Lambda_2 V = \binom{d}{2}$.

The inner product inherited from $T_2 V$ becomes

$$(\mathbf{x} \wedge \mathbf{y}, \mathbf{u} \wedge \mathbf{v}) = \frac{1}{2}((\mathbf{x}, \mathbf{u})(\mathbf{y}, \mathbf{v}) - (\mathbf{x}, \mathbf{v})(\mathbf{y}, \mathbf{u})) = \frac{1}{2} \det \begin{pmatrix} (\mathbf{x}, \mathbf{u}) & (\mathbf{x}, \mathbf{v}) \\ (\mathbf{y}, \mathbf{u}) & (\mathbf{y}, \mathbf{v}) \end{pmatrix}.$$

An orthonormal basis for $\Lambda_2 V$ is provided by $\{\sqrt{2}(e_i \wedge e_j) \mid 1 \leq i < j \leq d\}$.

Likewise, for $0 \leq k \leq d$ we define

$$\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k := \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \mathbf{x}_{\sigma 1} \otimes \dots \otimes \mathbf{x}_{\sigma k},$$

where $(-1)^\sigma$ is 1 (-1) if the permutation σ is even (odd), and γ_k is the symmetric group on k letters, and

$$\Lambda_k V := \langle \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k \mid \mathbf{x}_i \in V \rangle_R,$$

with inner product

$$(\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k) = \frac{1}{k!} \det((\mathbf{x}_i, \mathbf{y}_j)).$$

Note that $\dim \Lambda_k V = \binom{d}{k} = e_k(j)$; an orthonormal basis for $\Lambda_k V$ is provided by

$$\{\sqrt{k!} e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq d\}.$$

The exterior algebra ΛV is the (finite) direct sum

$$\Lambda V = \Lambda_0 V \oplus \Lambda_1 V \oplus \dots \oplus \Lambda_{d-1} V \oplus \Lambda_d V,$$

where $\Lambda_0 V \cong \Lambda_d V \cong R$ and $\Lambda_1 V \cong \Lambda_{d-1} V \cong V$. Its Poincaré polynomial is

$$p_\Lambda(t) = (1 + t)^d = E(t, j).$$

Remark

$\mathbf{x}_1, \dots, \mathbf{x}_k$ are linearly dependent iff $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k = \mathbf{0}$.

4. The symmetric algebra ΣV ^{5,6)}

$\Sigma_2 V$ is the linear space spanned by the symmetric 2-tensors,

$$\mathbf{x} \vee \mathbf{y} := \frac{1}{2}(\mathbf{x} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{x}); \quad \mathbf{x}, \mathbf{y} \in V.$$

Clearly, $\mathbf{x} \vee \mathbf{y} = \mathbf{y} \vee \mathbf{x}$ and $\dim \Sigma_2 V = \binom{d+1}{2}$.

$$(\mathbf{x} \vee \mathbf{y}, \mathbf{u} \vee \mathbf{v}) = \frac{1}{2}((\mathbf{x}, \mathbf{u})(\mathbf{y}, \mathbf{v}) + (\mathbf{x}, \mathbf{v})(\mathbf{y}, \mathbf{u})) = \frac{1}{2} \text{per} \begin{pmatrix} (\mathbf{x}, \mathbf{u}) & (\mathbf{x}, \mathbf{v}) \\ (\mathbf{y}, \mathbf{u}) & (\mathbf{y}, \mathbf{v}) \end{pmatrix}.$$

Likewise, for $0 \leq k < \infty$ we define

$$\Sigma_k V := \langle x_1 \vee \dots \vee x_k \mid x_i \in V \rangle_R,$$

where

$$x_1 \vee \dots \vee x_k = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} x_{\sigma 1} \otimes \dots \otimes x_{\sigma k}.$$

Then $\Sigma_k V$ has basis $\{e_{i_1} \vee \dots \vee e_{i_k} \mid 1 \leq i_1 \leq \dots \leq i_k \leq d\}$, and

$$\dim \Sigma_k V = \binom{d+k-1}{k} = h_k(j).$$

The inner product derives from

$$(x_1 \vee \dots \vee x_k, y_1 \vee \dots \vee y_k) = \frac{1}{k!} \text{per}((x_i, y_j)).$$

The symmetric algebra $\Sigma V = \sum_{k \geq 0}^{\oplus} (\Sigma_k V)$ has Poincaré polynomial

$$p_{\Sigma}(t) = (1-t)^{-d} = H(t, j).$$

Remark

The basis for $\Sigma_k V$ may be normalized using the factors $\left(\frac{k!}{k_1! \dots k_d!}\right)^{\frac{1}{2}}$, where $k! = k_1! \dots k_d!$ and $|k| = k_1 + \dots + k_d = k$, $e_{i_1} \vee \dots \vee e_{i_k} = e_1^{k_1} \vee \dots \vee e_d^{k_d}$.

5. Relating \mathcal{A} to Σ ^{7,8)}

We shall consider power series whose coefficients are vector spaces, with addition defined as direct sum, and multiplication as tensor product.

$$\mathcal{A}(t; V) := \sum_{k \geq 0} (\mathcal{A}_k V) t^k; \quad \Sigma(t; V) := \sum_{k \geq 0} (\Sigma_k V) t^k.$$

Theorem 5.1

$$\mathcal{A}(t; V \oplus W) = \mathcal{A}(t; V) \otimes \mathcal{A}(t; W); \quad \Sigma(t; V \oplus W) = \Sigma(t; V) \otimes \Sigma(t; W).$$

Proof

By considering bases we prove $\mathcal{A}_k(V \otimes W) = \sum_{i+j=k}^{\oplus} \mathcal{A}_i(V) \otimes \mathcal{A}_j(W)$, and the analogous formula for $\Sigma_k(V \oplus W)$.

These imply the relations of the theorem. □

Theorem 5.2

$$\mathcal{A}(-t; V) \otimes \Sigma(t; V) = 1.$$

Proof

First observe that for $\dim V = 1$ we have

$$\mathcal{A}(t; V) = 1 + tV; \quad \Sigma(t; V) = 1 + tV + t^2V + \dots = (1 - tV)^{-1}.$$

For an arbitrary V (which is the direct sum of one-dimensional subspaces), the theorem follows by application of theorem 5.1. \square

6. Relating Λ and Σ to S ^{4,8)}

The semigroup $(N, +)$ can be extended to the ring $(Z, +, \cdot)$ by a well-known construction. Similarly, the isomorphism classes $[V]$ of vector spaces V , over a fixed field form a semigroup with respect to direct sums, which can be extended to the Grothendieck ring K , with operations direct sum and tensor product. The map $[V] \rightarrow \dim V$ identifies the rings K and Z . For V of dimension d we have

$$[\Lambda_k V] \rightarrow \binom{d}{k} \quad \text{and} \quad [\Sigma_k V] \rightarrow \binom{d+k-1}{k}.$$

More interesting is the isomorphism of the Grothendieck ring K and the ring S of symmetric functions (in infinitely many variables). Here we have the following correspondences: (V of dimension d , $\mathbf{x} = (x_1, \dots, x_d, 0, 0, \dots)$)

$$\begin{aligned} \Lambda_k V &\rightarrow e_k(\mathbf{x}); & \Lambda(t; V) &\rightarrow E(t; \mathbf{x}), \\ \Sigma_k V &\rightarrow h_k(\mathbf{x}); & \Sigma(t; V) &\rightarrow H(t; \mathbf{x}). \end{aligned}$$

This shows the connection between theorems 5.2 and 1.1.

Remark

The ring S of the symmetric functions is the underlying ring for the representations of the symmetric group \mathcal{S} in the following sense: Let $\pi: \mathcal{S}_k \rightarrow \text{Aut}(V_\pi)$ denote an irreducible representation of \mathcal{S}_k on the \mathcal{S} -module V_π . Let $R(\mathcal{S})$ denote the ring of all such \mathcal{S} -modules. The fundamental theorem [5] says that there is an isomorphism (of λ -rings)

$$\Theta: R(\mathcal{S}) \simeq S.$$

We indicate how to obtain the symmetric function $\Theta(V_\pi)$: Let X be any vector space, consider $V_\pi \otimes X^{\otimes k}$ and its \mathcal{S}_k -invariant subspace $\pi(X)$. For instance, if $\pi = 1$ then $\pi(X) = \Sigma_k(X)$, if $\pi = +/ -$ then $\pi(X) = \Lambda_k(X)$. Any linear $T: X \rightarrow X$ induces $T_k(T): T_k X \rightarrow T_k X$ (see sec. 9) and $\pi(T): \pi(X) \rightarrow \pi(X)$. Now the function $\text{spec}(T) \rightarrow \text{trace } \pi(T)$ is the desired symmetric function in S which is attached to π .

7. Duality ⁶⁾

The dual V^* of the vector space V consists of the linear functionals on V . The action of V^* on V may conveniently be described by a pairing $[V, V^*]: [\mathbf{v}, \mathbf{w}^*] := \mathbf{w}^*(\mathbf{v})$. If we identify $(T_k V)^*$ with $T_k(V^*)$ we may write

$$[v_1 \otimes \dots \otimes v_k, w_1^* \otimes \dots \otimes w_k^*] = \prod_{i=1}^k [v_i, w_i^*] = \prod_{i=1}^k w_i^*(v_i).$$

Likewise we have $(\Lambda_k V)^* \simeq \Lambda_k(V^*)$ and $(\Sigma_k V)^* \simeq \Sigma_k(V^*)$ with the induced pairings

$$[v_1 \wedge \dots \wedge v_k, w_1^* \wedge \dots \wedge w_k^*] = \frac{1}{k!} \det (w_j^*(v_i)),$$

$$[v_1 \vee \dots \vee v_k, w_1^* \vee \dots \vee w_k^*] = \frac{1}{k!} \text{per} (w_j^*(v_i)).$$

Remark

ΣV^* is isomorphic with the ring of polynomial functions on V , with ordinary multiplication. In a certain sense ΛV^* corresponds to the set of square-free polynomials in d variables, cf. sec. 9.

8. Metrics⁶⁾

Let V have basis e_1, \dots, e_d and let e_1^*, \dots, e_d^* be the dual basis for V^* , i.e., $[e_i, e_j^*] = \delta_{ij}$.

In V a non-degenerate inner product is defined by a non-singular map $G: V \rightarrow V^*$ and $(v, w) := [v, Gw]$, $v, w \in V$. For the orthogonal geometry $O(r, s)$ the standard matrix for G , with respect to the bases e_i and e_j^* is

$$G = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}, \quad \text{and} \quad G = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

for the symplectic geometry. The map $G: V \rightarrow V^*$ induces $T_k G: T_k V \rightarrow T_k V^*$: $x_1 \otimes \dots \otimes x_k \rightarrow G x_1 \otimes \dots \otimes G x_k$, which in $T_k V$ leads to the inner product

$$(t, u) := [t, T_k G u]; \quad t, u \in T_k V,$$

or

$$(x_1 \otimes \dots \otimes x_k, y_1 \otimes \dots \otimes y_k) = \prod_{i=1}^k [x_i, G y_i] = \prod_{i=1}^k (x_i, y_i).$$

Likewise the maps $\Lambda_k G: \Lambda_k V \rightarrow \Lambda_k V^*$ and $\Sigma_k G: \Sigma_k V \rightarrow \Sigma_k V^*$ are defined and

$$(x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_k) = \frac{1}{k!} \det ((x_i, y_j)),$$

$$(x_1 \vee \dots \vee x_k, y_1 \vee \dots \vee y_k) = \frac{1}{k!} \text{per} ((x_i, y_j)).$$

9. Exterior and symmetric powers of a matrix^{5,6,9,10)}

Let $A: V \rightarrow V$ denote a linear map of V . The k -th tensor power $T_k A$ is defined by $T_k A: x_1 \otimes \dots \otimes x_k \rightarrow A x_1 \otimes \dots \otimes A x_k$. Likewise the maps $\Lambda_k A$ and $\Sigma_k A$ are defined.

Notation:

For k and l , with $|k| = |l| = k$ the matrix $A(k, l)$ is the $k \times k$ matrix which is obtained from (the $d \times d$ matrix) A by repeating k_i times the i -th row, and l_j times the j -th column, for $i, j = 1, \dots, d$.

Lemma 9.1

In $(Ax)^k$ the coefficient of x^l equals $\frac{1}{l!}$ per $A(k, l)$.

Proof

$(Ax)^k = \prod_{i=1}^d \left(\sum_{j=1}^d a_{ij} x_j \right)^{k_i}$. Arrange the terms $a_{ij} x_j$ in a $k \times d$ matrix. Change this into a $k \times k$ matrix with columns corresponding to l . Suppress x^l , then we are left with $A(k, l)$ and the result follows. \square

In the final remark of sec. 7 we mentioned that $\Lambda_k V$ corresponds in a certain sense to the square-free polynomials, the sense being that

$$x_{i_1} x_{i_2} \dots x_{i_k} = (-1)^\sigma x_{i_1} \dots x_{i_k}$$

for $\sigma \in \mathcal{G}_k$, i.e., the polynomials are *skew*. In this sense one has the following skew version of lemma 9.1:

Lemma 9.1'

In $(Ax)^k$ the coefficient of x^l equals $\det A(k, l)$. Note that $\det A(k, l) \neq 0$ implies $l! = 1$. By use of the orthonormal bases for $\Lambda_k V$ and $\Sigma_k V$ introduced in secs 3 and 4, we can now calculate the entries of the power matrices. \square

Theorem 9.2

$$\Lambda_k(A)_{k,l} = \det A(k, l); \quad \Sigma_k(A)_{k,l} = \frac{\text{per } A(k, l)}{\sqrt{k!} \sqrt{l!}}. \quad \square$$

From the definition we infer:

Theorem 9.3

Let A have eigenvalues $\alpha_1, \dots, \alpha_d$. The eigenvalues of $\Lambda_k(A)$ are the $\binom{d}{k}$ squarefree, those of $\Sigma_k(A)$ the $\binom{d+k-1}{k}$ homogeneous monomials of degree k in $\alpha_1, \dots, \alpha_d$. \square

The following expansion of the determinant is well-known:

$$\det(I + tA) = 1 + t \sum_{i=1}^d a_{ii} + t^2 \sum_{i < j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} + \dots + t^d \det(A).$$

Its counterpart for permanents is used less frequently:

Theorem 9.4

$$\det(I + tA) = \sum_{k=0}^d t^k \cdot \text{trace } \Lambda_k(A) = \sum_k t^{|\mathbf{k}|} \det A(\mathbf{k}, \mathbf{k});$$

$$\det^{-1}(I - tA) = \sum_{k \geq 0} t^k \cdot \text{trace } \Sigma_k(A) = \sum_k t^{|\mathbf{k}|} \frac{\text{per } A(\mathbf{k}, \mathbf{k})}{k!}.$$

Proof

The equalities on the right hand sides follow from theorem 9.2. As for the left hand sides, theorem 9.3 implies that $\text{trace } \Sigma_k(A)$ is the k -th complete symmetric polynomial in the eigenvalues α of A , i.e., we have

$$\text{trace } \Sigma_k(A) = h_k(\alpha), \quad \text{similarly } \text{trace } \Lambda_k(A) = e_k(\alpha).$$

Hence the sums of the traces are the generating functions $E(t; \alpha)$ and $H(t; \alpha)$ of sec. 1. The theorem now follows from theorem 1.1. \square

Corollary 9.5

$$\det(I + tA) = \text{trace } \Lambda(t; A) = \text{trace}^{-1} \Sigma(t; A),$$

for the generating functions

$$\Lambda(t; A) = \sum_{k \geq 0} t^k \Lambda_k(A) \quad \text{and} \quad \Sigma(t; A) = \sum_{k \geq 0} t^k \Sigma_k(A). \quad \square$$

10. Applications

I. G. Macdonald told us the following proof of MacMahon's master theorem.

Theorem 10.1

The coefficient of $\mathbf{x}^{\mathbf{k}}$ in the symmetric product $(A \mathbf{x})^{\mathbf{k}}$ equals the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $1/\det(I - A \Delta(\mathbf{x}))$, where $\Delta(\mathbf{x}) = \text{diag}(x_1, \dots, x_d)$.

Proof

By 9.1 and 9.2 the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $(A \mathbf{x})^{\mathbf{k}}$ equals the (\mathbf{k}, \mathbf{k}) entry of $\Sigma_{\mathbf{k}}(A)$ where $\mathbf{k} = |\mathbf{k}|$. Hence it is the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $\text{trace } \Sigma_{\mathbf{k}}(A \Delta(\mathbf{x}))$. However, $\sum_{k \geq 0} \text{trace } \Sigma_k(A \Delta(\mathbf{x})) = 1/\det(I - A \Delta(\mathbf{x}))$, according to theorem 9.4. \square

Theorem 10.2¹⁾

$$\frac{(\mathbf{x}, A \mathbf{y})^{\mathbf{k}}}{k!} = \sum_{|\mathbf{k}|=|\mathbf{l}|=k} \frac{\mathbf{x}^{\mathbf{k}} \mathbf{y}^{\mathbf{l}}}{k! \mathbf{l}!} \text{per } A(\mathbf{k}, \mathbf{l}).$$

Proof

Write $\mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_d \mathbf{e}_d$, and take the inner products of the symmetric tensors $\frac{1}{k!} \mathbf{x} \vee \dots \vee \mathbf{x}$ and $\frac{1}{k!} A \mathbf{y} \vee \dots \vee A \mathbf{y}$, using

$$\frac{1}{k!} x \vee \dots \vee x = \sum_{|k|=k} \frac{x^k}{k!} e_1^{k_1} \vee \dots \vee e_d^{k_d}. \quad \square$$

Corollary 10.3 (N. G. de Bruijn)

For any rectangular matrix $A = (a_{ij})$ we have

$$\frac{1}{k!} (\sum_{i,j} a_{ij})^k = \sum_{|k|=|l|=k} \frac{\text{per } A(k, l)}{k! l!}.$$

Proof

Make A into a square matrix by supplying zeros, and apply theorem 10.2. □

11. Fredholm's integral equation^{2,3,11}

From the theory of integral equations it follows that

$$u(x) = f(x) + \lambda \int_0^1 K(x, t) u(t) dt$$

has the solution

$$u(x) = f(x) + \int_0^1 \frac{D(x, y; \lambda)}{D(\lambda)} f(y) dy,$$

provided $D(\lambda) \neq 0$. To explain the notation, we divide the interval $[0, 1]$ into d equal parts by $0 < \frac{1}{d} < \dots < \frac{d-1}{d} < 1$. Let f be the d -vector with components $f_i = f(\frac{i}{d})$, and let M be the $d \times d$ matrix with entries $M_{ij} = \frac{1}{d} K(\frac{i}{d}, \frac{j}{d})$. Then the integral equation is approximated by the following set of matrix equations of increasing size $d = 1, 2, 3, \dots$

$$(I - \lambda M)u = f.$$

These equations may be solved using Cramer's rule. Fredholm's determinant $D(\lambda)$ is defined by

$$1 - \lambda \int_0^1 K(t, t) dt + \frac{\lambda^2}{2!} \int_0^1 \int_0^1 \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) \\ K(t_2, t_1) & K(t_2, t_2) \end{vmatrix} dt_1 dt_2 - \frac{\lambda^3}{3!} \dots$$

It is the limit, for $d \rightarrow \infty$, of

$$\Delta(\lambda) := 1 - \lambda \sum_i M_{ii} + \frac{\lambda^2}{2!} \sum_{i,j} \begin{vmatrix} M_{ii} & M_{ij} \\ M_{ji} & M_{jj} \end{vmatrix} + \dots + (-1)^d \lambda^d \det M.$$

Fredholm's first minor $D(x, y; \lambda)$ is defined by

$$\lambda K(x, y) - \lambda^2 \int_0^1 \begin{vmatrix} K(x, y) & K(x, t) \\ K(t, y) & K(t, t) \end{vmatrix} dt + \frac{\lambda^3}{2!} \int_0^1 \int_0^1 \begin{vmatrix} K(x, y) & K(x, s) & K(x, t) \\ K(s, y) & K(s, s) & K(s, t) \\ K(t, y) & K(t, s) & K(t, t) \end{vmatrix} ds dt + \dots$$

It is the limit for $d \rightarrow \infty$, of

$$\Delta(x, y; \lambda) := \lambda M_{xy} - \lambda^2 \sum_i \begin{vmatrix} M_{xy} & M_{xt} \\ M_{iy} & M_{it} \end{vmatrix} + \frac{\lambda^3}{2!} \sum_{i,j} \begin{vmatrix} M_{xy} & M_{xt} & M_{xj} \\ M_{iy} & M_{it} & M_{ij} \\ M_{jy} & M_{ji} & M_{jj} \end{vmatrix} + \dots$$

By replacing determinants by permanents we get expressions which we denote by $P(-\lambda)$, $\Pi(-\lambda)$, $-P(x, y; -\lambda)$, $-\Pi(x, y; -\lambda)$, respectively. Now in ref. 2 it is proved that

$$D(\lambda) P(\lambda) = 1, \quad D(x, y; \lambda) P(\lambda) = P(x, y; \lambda) D(\lambda). \quad (*)$$

As a consequence the solution of Fredholm's equation may also be written in terms of permanents:

$$u(x) = f(x) + \int_0^1 \frac{P(x, y; \lambda)}{P(\lambda)} f(y) dy.$$

Now the first equality in (*) is implied by

$$\Delta(\lambda) = \det(I - \lambda M) = (\Pi(\lambda))^{-1}.$$

One can verify that $\Delta(x, y; \lambda) \Pi(\lambda) = \Pi(x, y; \lambda) \Delta(\lambda)$ also holds, but we lack an elegant proof of this fact.

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